JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **141**, 399–404 (1989)

A Formula for the Mass Density of a Vibrating String in Terms of the Trace

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Submitted by V. Lakshmikantham

Received November 18, 1986

We consider the equation of the vibrating string

$$u'' + \lambda \rho u = 0 \qquad \text{on} \quad 0 \leqslant x \leqslant 1 \quad \text{with } u(0) = 0 = u(1). \tag{1}$$

Here ρ is positive, has a continuous first derivative, and has a second derivative in L^2. We denote this class of functions by H^2_+. This problem has a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$. In a previous paper the author developed some interesting consequences which arise on consideration of the eigenvalues as functions of the length of the interval. In particular, if we solve (1) with the condition at the right end replaced by $u(x) = 0$, then we have a sequence of eigenvalues $0 < \lambda_1(x) < \lambda_2(x) < \cdots$. These functions are solutions of $U(x, \lambda) = 0$, where $U(x, \lambda)$ solves Eq. (1) with the initial conditions $U(0, \lambda) = 0$ and $U'(0, \lambda) = 1$. Therefore the functions $\lambda_n(x)$ have continuous second derivatives. They are also decreasing functions of x. The results of this previous paper which are relevant here are Theorem 1 and Proposition 1 below. The main result of that paper may be stated by saying that the pair of sequences $(\lambda_k(1), \lambda_k'(1))$ suffices as data to guarantee the existence and uniqueness of the function ρ in (1); i.e., the pair of sequences referred to enables one to solve the inverse problem for (1) uniquely.

In this paper we go a step further and prove that the function ρ can be obtained by the formula

$$\rho = \frac{d^2\,\mathrm{tr}}{dx^2} + \frac{2}{x}\frac{d\,\mathrm{tr}}{dx}, \tag{2}$$

where $\mathrm{tr}(x) = \sum (\lambda_k(x))^{-1}$. This formula shows that the function ρ is determined by the eigenvalues and their rates of change as well.

It is important, for what follows, to know the asymptotic form of the eigenvalues and their derivatives. We work with the frequencies $\omega_n^2 = \lambda_n$.

0022-247X/89 $3.00

PROPOSITION 1. *If* $\rho \in H^2_+$, *then*

$$\omega_n = an + b/n + a_n, \qquad \text{where} \quad 0 < \omega_1 < \omega_2 < \cdots, \quad a > 0,$$
$$\text{and} \quad (a_n) \in l^2_1, \quad \text{i.e., } \Sigma(na_n)^2 < \infty,$$
$$d\omega_n/dx = cn + d/n + c_n, \qquad \text{where} \quad c < 0, \omega_n' < 0 \quad \text{for every } n,$$
$$\text{and} \quad (c_n) \in l^2.$$

Furthermore, as functions of x *the estimates hold uniformly for* x *in a compact subset of* $(0, 1]$.

One of the most interesting features of this approach to the study of eigenvalues is that (1) and the form which $U(x, \lambda)$ takes, viz., $U(x, \lambda) = x\Pi(1 - \lambda/\lambda_k(x))$, work together to give a system of coupled equations;

THEOREM 1. *If* $\rho \in H^2_+$, *then*

$$\lambda_n'' + (2/x)\lambda_n' + 2\lambda_n\lambda_n' S_n - 2(\lambda_n')^2/\lambda_n = 0, \qquad n = 1, 2, ..., \tag{3}$$

where $S_n = \Sigma(\lambda_k'/\lambda_k)(\lambda_k - \lambda_n)^{-1}$ *and* $k \neq n$ *in the sum.*

It is this system of equations which leads to the formula for ρ. The proofs of Proposition 1 and Theorem 1 appear in [5]. The proof of Theorem 1 follows by differentiating twice with respect to x in $U(x, \lambda_n(x)) = 0$, using (1) and the form of U as a product. The various term by term differentiations of sums and products are justified by the conclusion of Proposition 1. We indicate the idea of the proof of Proposition 1. The Liouville transformation which takes (x, λ, ρ) into (t, Λ, Q) maps $\rho \in H^2_+$ into $Q \in L^2$, where

$$T = \int_0^1 \sqrt{\rho}\, ds, \qquad t = \int_0^x \sqrt{\rho}\, ds/T, \qquad \Lambda = T^2\lambda$$

and

$$Q(t) = -3/16(R^{-1}\, dR/dt)^2 + 1/4(R^{-1}\, d^2R/dt^2) \qquad \text{and}$$
$$R(t) = \rho(x).$$

If we set $Y(t, \Lambda, Q) = (\rho(x)\rho(0))^{1/4}\, U(x)/T$ then (1) takes the canonical form $Y'' + (\Lambda - Q)Y = 0$ and $Y(1, \Lambda) = 0$ gives the eigenvalues. We use this well-known map because the asymptotics for the canonical case are readily available. Then

$$\Lambda_k = T^2\lambda_k, \qquad d\Lambda_k/dt = (T^3/\sqrt{\rho})\, d\lambda_k/dx$$

and

$$dA_k/dt(1) = -(\partial Y/\partial t(1, A_k))^2 \Big/ \left(\int_0^1 Y^2(t, A_k)\, dt \right).$$

These relations and the estimates of [4, Chap. 2] work to estimate $d\lambda_k/dt$. The estimate for λ_k comes from that of A_k (also in [4]).

The formula for ρ in terms of the trace involves the second derivatives λ_n''. This means that the estimates of Proposition 1 must be extended to include ω_n'' or what is the same thing λ_n''.

PROPOSITION 2. *If $\rho \in H^2_+$, then λ_n'' is $O(n^2)$ uniformly for x in a compact subset of* $(0, 1]$.

The proof of this uses the coupled equations (3). We use a prime on a summation to indicate that the term for $k = n$ is omitted.

LEMMA 1.

$$\sum{}' (k^2 - n^2)^{-2} = O(1/n^2) \tag{a}$$

$$\sum{}' k^{-2}(k^2 - n^2)^{-2} = O(1/n^4). \tag{b}$$

Proof. The sum in (a) is

$$(1/4n^2)\left[\sum{}' (k-n)^{-2} - 2\sum{}' (k^2 - n^2)^{-1} + \sum{}' (k+n)^2\right]$$

and each sum on the right is bounded in n. The sum in (b) is

$$(1/n^4)\left[\sum{}' k^{-2} - 2\sum{}' (k^2 - n^2)^{-1} + \sum{}' (k/(k^2 - n^2))^2\right]$$

and again each sum on the right is bounded in n.

Proof of Proposition 2. The use of Theorem 1 and Proposition 1 shows that it is enough to prove that $S_n = O(1/n^2)$ uniformly for x in a compact subset of $(0, 1]$. The estimates for ω_n and ω_n' show that

$$\lambda_n'/\lambda_n = 2\omega_n'/\omega_n = c + b_k/k, \qquad \text{where} \quad (b_k) \in l^2.$$

Therefore,

$$S_n = c\sum{}' (\lambda_k - \lambda_n)^{-1} + \sum{}' (b_k/k)(\lambda_k - \lambda_n)^{-1}.$$

Sums similar to this are treated in [2, pp. 34–35]. It is important to avoid the use of absolute values in these sums since this will change the order of magnitude. We can assume that $\lambda_k = k^2 + a_k$, where $(a_k) \in l^2$. We have

$$\begin{aligned}\lambda_k - \lambda_n &= (k^2 - n^2) + (a_k - a_n)\\ &= (k^2 - n^2)[1 + (a_k - a_n)/(k^2 - n^2)],\end{aligned}$$

so

$$\begin{aligned}1/(\lambda_k - \lambda_n) = 1/(k^2 - n^2) &- (a_k - a_n)/(k^2 - n^2)^2\\ &+ (a_k - a_n)^2/\{(k^2 - n^2)^2(\lambda_k - \lambda_n)^2\};\end{aligned}$$

hence

$$\begin{aligned}\sum{}' (\lambda_k - \lambda_n)^{-1} = \sum{}' (k^2 - n^2)^{-1} &- \sum{}' (a_k - a_n)/(k^2 - n^2)^2\\ &+ \sum{}' (a_k - a_n)^2/\{(k^2 - n^2)(\lambda_k - \lambda_n)\}.\end{aligned}$$

Now we make use of the lemma. The first sum is $3/(4n^2)$, the second is $O(1/n^2)$, and the third is $0(1/n^3)$. Now we treat

$$\begin{aligned}&\sum{}' b_k/\{k(\lambda_k - \lambda_n)\}\\ &\quad = \sum{}' b_k/\{k(k^2 - n^2)\} - \sum{}' \{b_k(a_k - a_n)\}/\{k(k^2 - n^2)\}\\ &\quad + \sum{}' \{b_k(a_k - a_n)^2\}/\{k(k^2 - n^2)^2(\lambda_k - \lambda_n)\}.\end{aligned}$$

The second and third sums on the right are $O(1/n^2)$. We treat the first as the inner product of two l^2 sequences and have

$$\begin{aligned}&\left|\sum{}' b_k/\{k(k^2 - n^2)\}\right|\\ &\quad \leqslant \left[\sum b_k\right]^{1/2} \left[\sum{}' k^{-2}(k^2 - n^2)^{-2}\right]^{1/2} = O(1/n^2)\end{aligned}$$

by the result of the lemma. Therefore we have shown that $S_n = O(1/n^2)$. The estimates for S_n hold uniformly for x in a compact subset of $(0, 1]$ since this is the case for the estimates for λ_n and λ_n'.

LEMMA 2. *If* $\rho \in H^2_+$, *then*

$$d\,\mathrm{tr}/dx = -\sum \lambda'_k/\lambda_k^2,$$

$$d^2\,\mathrm{tr}/dx^2 = -\sum [\lambda''_k/\lambda_k^2 - 2(\lambda'_k)^2/\lambda_k^3];$$

i.e., term by term differentiation of these sums is justified.

The proof follows directly from the uniform estimates for the quantities λ_n, λ'_n, and λ''_n.

THEOREM 2. *If* $\rho \in H^2_+$, *then for* $0 < x \leqslant 1$

$$\rho(x) = \sum 2(\lambda'_k/\lambda_k) S_k.$$

Proof. The function $U(x, \lambda)$ is an entire function of λ. Since $\partial^2 U/\partial x^2 = -\lambda \rho U$, $\partial^2 U/\partial x^2$ is also an entire function of λ. Since $U(x, 0) = x$ it follows that the coefficient of λ in the power series expansion of $\partial^2 U/\partial x^2$ is $-x\rho(x)$. Now we make use of the form of U, viz., $U = x\Pi(1 - \lambda/\lambda_k(x))$. We differentiate twice with respect to x and use the coupled equations (3). Hence,

$$\frac{\partial^2 U}{\partial x^2} = -\lambda x \sum [(\lambda'_n/\lambda_n)(2F_n S_n) - (\lambda'_n/\lambda_n^2)\partial F_n/\partial x],$$

where $F_n(x, \lambda) = \Pi(1 - \lambda/\lambda_k)$ and the product is taken over those $k \neq n$. Then ρ is the quantity in the above sum when that sum is evaluated at $\lambda = 0$. But $F_n(x, 0) = 1$ and $\partial F_n/\partial x\ (x, 0) = 0$ for every n so

$$\rho(x) = 2 \sum (\lambda'_n/\lambda_n) S_n.$$

The differentiation of the various sums is justified on the basis of the properties of λ_k, λ'_k, λ''_k.

COROLLARY 1. *If* $\rho \in H^2_+$, *then for* $0 < x \leqslant 1$

$$\rho = \frac{d^2\,tr}{dx^2} + \frac{2}{x}\frac{d\,\mathrm{tr}}{dx}.$$

This follows directly from the previous theorem, the coupled equations, and Lemma 2.

Another useful formula for ρ follows on differentiation of Liouville's formula

$$\lim_n (n\pi/\omega_n(x)) = \int_0^x \sqrt{\rho}\, ds.$$

We expect

$$\lim_n [(-n\pi/\omega_n)\omega_n'/\omega_n] = \sqrt{\rho}.$$

This holds for every x on $(0, 1]$ and can be proved on the basis of the uniform estimates for ω_n and ω_n'.

We observe here again that the sequences $(\lambda_n(1)), (\lambda_n'(1))$ uniquely determine $\lambda_n(x)$, therefore uniquely determine the trace and hence $\rho(x)$ [5].

The formula for the function ρ in terms of the derivatives of the trace indicates the likelihood of finding other valuable relations involving the zeta function $\Sigma\omega_n^{-z}(x)$ and the function ρ. We note in passing that there is another formula for ρ which involves the sequence (λ_n) and the eigenvalues (μ_n), where μ_n arises as a solution to Eq. (1) with the condition $u'(0) = u(1) = 0$. This forms the basis of the paper [1]. Also, for the benefit of the reader we include reference to a recent survey article [3] which discusses the inverse Sturm–Liouville problem in general.

It is clear that this trace formula is valid under less restrictive conditions on ρ. Any assumptions on ρ which lead to the uniform estimates in Proposition 1 and the uniform estimate $S_n = O(1/n^2)$ will yield the same result. Also observe that if we specialize to $\rho = 1$, then $\lambda_n(x) = n^2\pi^2/x^2$, $\mathrm{tr}(x) = (x^2/\pi^2)\Sigma(1/n^2)$, and formula (2) gives a proof that $\Sigma(1/n^2) = \pi^2/6$.

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